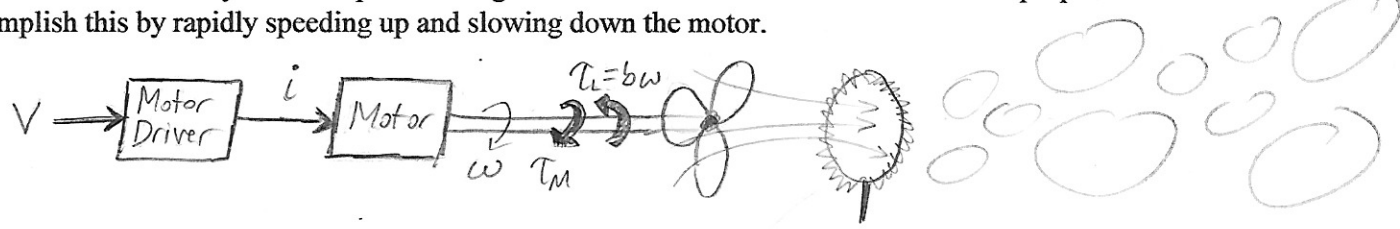
Mechanical Systems Laboratory, April 16, 2016
Proportional Feedback Control of Motor Velocity (used in Lab 4)

1. Motor Dynamics

Next week in lab, you will use an Arduino equipped with a **motor driver** shield to control the velocity of a DC motor using **proportional feedback**. To place this lab in "context" imagine you are designing a control system for a sophisticated Bunches of Bubble ("Bob") machine. Bob uses a motor-driven fan to blow bubbles. Turning the fan slowly produces large bubbles while turning the fan quickly produces lots of small bubbles. Ideally Bob will produce large and small bubbles at the same time. You propose to accomplish this by rapidly speeding up and slowing down the motor.



The motor driver powers the motor using an input current, i, proportional to a voltage, V, generated by the Arduino. The motor produces a torque, τ_M, proportional to i. τ_M causes the speed of the motor, $\omega(t)$, to increase at a rate proportional to the rotating inertia, J, until the motor becomes loaded by an equal and opposite torque modelled as viscous friction, $\tau_L = b\omega(t)$. Thus the model of the **open loop** system is:

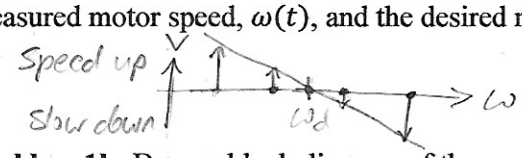
$$\tau_M = \alpha V$$
$$\tau_M = J\dot{\omega} + b\omega$$
$$\Downarrow$$
$$J\dot{\omega}(t) + b\omega(t) = \alpha V(t)$$

Open Loop Transfer Function $G_o(s)$

$$(Js + b)\omega(s) = \alpha V(s)$$
$$\omega(s) = G_o(s)V(s)$$
$$G_o(s) = \frac{\omega(s)}{V(s)} = \frac{\alpha}{Js+b}$$

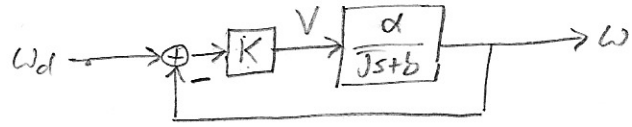
$$V \rightarrow \boxed{\frac{\alpha}{Js+b}} \rightarrow \omega$$

Problem 1a: Design a **proportional feedback control law** for the Arduino's voltage, V, based on the measured motor speed, $\omega(t)$, and the desired motor speed, $\omega_d(t)$.



$$V = -K(\omega - \omega_d)$$

Problem 1b: Draw a block diagram of the controlled system.



Problem 1c: Write the differential equation that describes the behavior of the controlled system.

$$J\dot{\omega} + b\omega = -\alpha K(\omega - \omega_d)$$
$$J\dot{\omega} + (b + \alpha K)\omega = \alpha K \omega_d$$

Problem 1d: Find the transfer function for the controlled (or "**closed loop**") system.

$$(Js + b + \alpha K)\omega = \alpha K \omega_d$$
$$G(s) = \frac{\omega}{\omega_d} = \frac{\alpha K}{Js + b + \alpha K}$$

2. Using a Transfer Function

Problem 2a: Convert the open and closed loop transfer functions into **canonical form** to find the time constants of the open and closed loop systems.

Canonical Form

$$G(s) = \frac{A}{\tau s + 1}$$

$$G_o(s) = \frac{\alpha}{Js+b} \times \frac{\frac{1}{b}}{\frac{1}{b}} = \frac{\frac{\alpha}{b}}{\left(\frac{J}{b}\right)s+1} \Rightarrow \tau_o = \frac{J}{b}$$

$$G(s) = \frac{\alpha K}{Js+b+\alpha K} \times \frac{\frac{1}{b+\alpha K}}{\frac{1}{b+\alpha K}} = \frac{\frac{\alpha K}{b+\alpha K}}{\left(\frac{J}{b+\alpha K}\right)s+1} \Rightarrow \tau = \frac{J}{b+\alpha K}$$

Increasing the proportional gain, K, ~~increases~~ **decreases** the time constant of the system, meaning that the system changes its speed, ω, more **quickly** ~~slowly~~ in response to changes in ω_d.

Problem 2b: Use the transfer function to determine the **steady state velocity**, ω_{ss}, and **steady state error**, e_{ss}, of the controlled system in response to a constant ω_d.

$$\omega_{ss} = G(0)\omega_d = \frac{\alpha K}{J(0)+b+\alpha K}\omega_d = \left(\frac{\alpha K}{b+\alpha K}\right)\omega_d$$

$$e_{ss} = \omega_{ss} - \omega_d = \left(\frac{\alpha K}{b+\alpha K}\right)\omega_d - \left(\frac{b+\alpha K}{b+\alpha K}\right)\omega_d = -\left(\frac{b}{b+\alpha K}\right)\omega_d$$

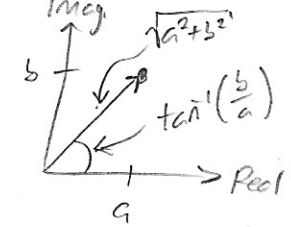
Increasing the proportional gain, K, ~~increases~~ **decreases** the steady state error.

Problem 2c: Use the transfer function to determine the speed of the motor, $\omega(t)$, in response to a sinusoidal desired speed input: $\omega_d(t) = A\sin(2\pi f t)$.

$$\omega(t) = \underbrace{|G(2\pi f j)|}_{\substack{\text{gain of } G(s) \\ \text{evaluated at} \\ s=2\pi f j}} A \sin\left(2\pi f t + \underbrace{\phi_{G(2\pi f j)}}_{\substack{\text{phase of } G(s) \\ \text{evaluated at} \\ s=2\pi f j}}\right)$$

Aside

For complex $a+bj$



$$a+bj = \underbrace{\sqrt{a^2+b^2}}_{\text{gain}}\, e^{j\underbrace{\tan^{-1}\left(\frac{b}{a}\right)}_{\text{phase}}}$$

$$G(2\pi f j) = \frac{\alpha K}{(2\pi f J)j + (b+\alpha K)}$$

$$G(2\pi f j) = \frac{N(2\pi f j)}{D(2\pi f j)} = \frac{A_N e^{j\phi_N}}{A_D e^{j\phi_D}} = \frac{A_N}{A_D} e^{j(\phi_N - \phi_D)}$$

$$|G(2\pi f j)| = \frac{\alpha K}{\sqrt{(b+\alpha K)^2 + (2\pi J)^2 f^2}} \qquad \phi_{G(2\pi f j)} = -\tan^{-1}\left(\frac{2\pi J f}{b+\alpha K}\right)$$

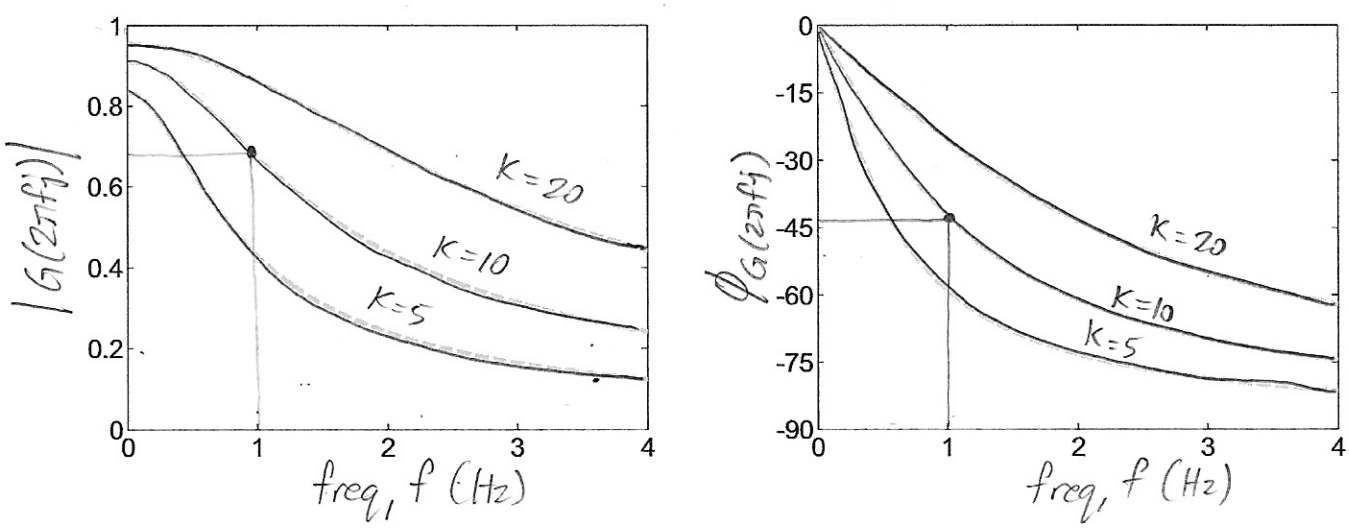
Problem 2d: Using a simplified case where $b = 1$, $\alpha = 1$ and $J = 10/2\pi$, show that the controlled system behaves like a **low-pass filter** by plotting the **gain** and **phase shift** of the system's response to sinusoidal inputs with frequencies ranging from 0 to 4 Hz. Plot the cases where $K = 5, 10$ and 20.

$$|G(2\pi fj)| = \frac{K}{\sqrt{(1+K)^2 + 100f^2}} \qquad \phi_{G(2\pi fj)} = -\tan^{-1}\left(\frac{10f}{1+K}\right)$$

let $K = 10$, $f = 1$ Hz

$$|G(2\pi j)| = \frac{10}{\sqrt{121+100}} = \frac{10}{\sqrt{221}} = 0.673$$

$$\phi_{G(2\pi j)} = -\tan^{-1}\left(\frac{10}{11}\right) = -0.74\,\text{rad} = -42^\circ$$



As the frequency of $\omega_d(t)$ increases, the amplitude of $\omega(t)$ ~~increases~~ **decreases.**

As the proportional control gain, K, increases, the amplitude of $\omega(t)$ **increases** ~~decreases.~~

As the frequency of $\omega_d(t)$ increases, the phase shift approaches 90 degrees ($\frac{\pi}{2}$ radians).

Conclusion: In order for Bob to produce bubbles of a wide range of sizes each second, where the largest bubbles are created when $\omega = 300\ RPM$ and the smallest bubbles are created when $\omega = 900\ RPM$, a good controller could be designed using $\omega_d(t) = 600\ RPM + (300\ RPM) * \sin(2\pi(1\ Hz)t)$, and $K = 20$. The resulting motor speed will be $\omega(t) = 571\ RPM + (257\ RPM) * \sin(2\pi(1\ Hz)t - 25°)$. This is because the constant term ($f = 0$ Hz) is attenuated by a gain of 0.95 and the 1 Hz term is attenuated by a gain of 0.86. What could you do to improve this performance?

Increase K
Change ω_d to make ω closer to 300 - 900RPM
Add Integral Control to remove e_{ss}